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The Effect of Repulsive Interactions on Bose–Einstein Condensation

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One-dimensional Bose gases that interact via a repulsive two-body interaction and show Bose–Einstein condensation at the free level are studied. It is shown that the introduction of this interaction, however small, destroys the condensate. It is also shown that the free energy of an interacting Bose gas does not depend on the boundary conditions *(including attractive boundary conditions)* in the van der Waals limit.

KEY WORDS: One-dimensional Bose gases; Bose–Einstein condensation; dependence of free energy on boundary conditions.

1. INTRODUCTION

An open problem in contemporary physics is whether Bose–Einstein condensation, which appears in free Bose gases,[1] persists when an interaction is switched on. While most physicists will agree that it does, no rigorous proof of this conjecture yet exists. Only in certain special cases have partial results been obtained.

(i) A mean field repulsive interaction does not destroy the condensate.[2–4] The condensate also persists in the van der Waals limit of a two-body interaction of positive type.[5]

(ii) If the Hamiltonian of the free gas has a gap in its spectrum, Bose–Einstein condensation is stable under perturbation by any integrable two-body potential of positive type.[6] However, the introduction of a gap in this paper is purely phenomological and is not motivated from first principles.

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There have been several results, though, on the absence of condensation.

(a) Using Bogoliubov's inequality, one can show that a Bose gas interacting through a superstable potential and with Neumann or Dirichlet boundary conditions at the walls of the container does not show condensation in one or two dimensions.[7,8]

(b) Lenard[9,10] and Schultz[11] have shown that there is no condensation (even in the generalized sense) in a one-dimensional model of bosons with point hard cores and periodic boundary conditions.

It should be noted, however, that, in both cases (a) and (b) the corresponding free model has no condensate either, and therefore the absence of condensation is not fully surprising.

(c) More recently, Buffet and Pulè[12,13] have shown that, in certain one-dimensional models that exhibit condensation at the free level, the condensate can be destroyed by introducing hard core interactions. Their results show how unstable Bose–Einstein condensation can be.

In this paper, similar models are considered when the hard core interaction is replaced by a small repulsive interaction. It is shown that, both in the case of a one-dimensional Bose system with attractive boundary conditions or in an external field, the condensate, if defined as in the free case, is destroyed by the introduction of the repulsive interaction, however small.

Finally, the free energy of the one-dimensional Bose gas with attractive boundary conditions is considered. It is generally expected that the free energy of a Bose system with any reasonable interaction does not depend on the boundary conditions. This has indeed rigorously been proved (e.g., Refs. 14 and 15) at low densities. Recently, Park[16] has been able to prove that the pressure is independent of the boundary conditions in the case of interactions that are strongly superstable and this at any density. His analysis, however, does not cover the case of attractive boundary conditions. It should be noted that also in the free model, the mean field model, as in the hard core model studied by Buffet and Pulè,[12] the free energy does depend strongly on the boundary conditions when attractive boundary conditions are considered. However, it will be shown that the conjecture holds in the van der Waals limit of a repulsive two-body potential of positive type. It remains an open but interesting question whether this result still holds when this limit is not taken.

2. THE MODEL

Consider a one-dimensional Bose system of N particles in a box $[0, L]$, with one-particle Hamiltonian h_1^L having eigenvalues ε_k^L and corresponding eigenfunctions f_k^L ($k=0, 1, 2,...$). The main interest of this paper will be the case

$$h_1^L = h_1^{L,\sigma} \equiv -\frac{1}{2}\frac{d^2}{dx^2}$$

with boundary conditions

$$\frac{d\psi}{dx}(x) = -\sigma\psi(x) \qquad \text{for} \quad x=0 \text{ and } x=L\ (\sigma>0) \tag{1}$$

(this corresponds to attractive boundary conditions at $x=0$ and repulsive boundary conditions at $x=L$). The eigenfunctions and eigenvalues of $h_1^{L,\sigma}$ are given by

$$f_0^{L,\sigma}(x) = \left[\frac{2\sigma}{1-\exp(-2\sigma L)}\right]^{1/2} \exp(-\sigma x), \qquad \varepsilon_0^{L,\sigma} = -\frac{\sigma^2}{2} \tag{2a}$$

$$f_n^{L,\sigma}(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L} + \alpha_n^{L,\sigma}\right), \qquad \varepsilon_n^{L,\sigma} = \frac{n^2\pi^2}{L^2} \quad (n \geqslant 1) \tag{2b}$$

where $\alpha_n^{L,\sigma}$ satisfies $\tan \alpha_n^{L,\sigma} = n\pi/\sigma L$.

The proof, presented below, allows one, moreover, to treat one-particle Hamiltonians of the form

$$h_1^L = -\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{x}{L}\right)^\alpha$$

with either Dirichlet or Neumann boundary conditions (V is an external potential).

The appropriate space to describe N bosons in the box $[0, L]$ is $\mathscr{H}_N^L \equiv S(L^2([0, L]; dx))^{\otimes N}$ where S is the usual symmetrization operator. The free Hamiltonian H_0^L related to h_1^L is then defined by

$$H_0^L \psi_1 \otimes \cdots \otimes \psi_N = h_1^L \psi_1 \otimes \cdots \otimes \psi_n + \cdots + \psi_1 \otimes \cdots \otimes h_1^L \psi_n$$

Note that H_0^L can conveniently be written as

$$H_0^L = \sum_{k \geqslant 0} \varepsilon_k^L a^*(f_k^L)\, a(f_k^L) = \sum \varepsilon_k^L N(f_k^L) \tag{3}$$

where $a^*(f)$ and $a(f)$ are the usual creation and annihilation operators and $N(f)=a^*(f)\,a(f)$.

Consider a two-body potential U satisfying

$$\text{(i)}\quad 0\leqslant U(x)<\infty,\qquad \forall x\in\mathbb{R} \tag{4a}$$

$$\text{(ii)}\quad \exists A>0,\ \exists\varepsilon>0 \text{ such that } \forall x\in[-A,+A]:\quad U(x)\geqslant\varepsilon \tag{4b}$$

$$\text{(iii)}\quad \int_{\mathbb{R}} dx\, U(x)<\infty \tag{4c}$$

The Hamiltonian of the interacting Hamiltonian is then given by

$$H^L=H_0^L+\mathcal{U} \tag{5a}$$

where

$$\mathcal{U}(x_1,\ldots,x_N)=\sum_{1\leqslant i<j\leqslant N} U(x_i-x_j)$$

Since $\mathcal{U}$ is a bounded operator [by (4)], H^L has the same domain as H_0^L, and H_0^L in (5a) can still be represented in the form (3). (This would not be the case if one were to consider hard core interactions.) The state ω^L is defined as

$$\omega^L(A)=\mathrm{Tr}_{\mathscr{H}_N^L} A\exp(-\beta H^L)/\mathrm{Tr}_{\mathscr{H}_N^L}\exp(-\beta H^L) \tag{5b}$$

One is then interested in the thermodynamic limit $L\to\infty$ such that

$$\lim_{L\to\infty}\frac{N}{L}=\rho$$

It is well known how the free model ($\mathcal{U}=0$) behaves in this limit. For example, in the case $h_1^L=h_1^{L,\sigma}$ (denote the corresponding state by $\omega_0^{L,\sigma}$), there exists a critical density

$$\rho_c^\sigma=\frac{1}{(2\pi\beta)^{1/2}}\,g_{\frac{1}{2}}(\exp-\tfrac{1}{2}\beta\sigma^2)<\infty\qquad \text{if}\quad \sigma>0$$

with

$$g_{\frac{1}{2}}(x)=\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$$

such that

$$\begin{aligned}&\text{(i)}\quad \text{if } \rho \leqslant \rho_c^\sigma: \quad \lim_{L\to\infty} \omega_0^{L,0} \frac{N(f_0^{L,0})}{L} = 0\\ &\text{(ii)}\quad \text{if } \rho > \rho_c^\sigma: \quad \lim_{L\to\infty} \omega_0^{L,\sigma} \frac{N(f_0^{L,\sigma})}{L} = \rho - \rho_c^\sigma\end{aligned} \tag{6}$$

[$f_0^{L,\sigma}$ as in (2)].

In the next section, it will be shown that if a two-body interaction U satisfying (4) is added, then

$$\lim_{L\to\infty} \omega^{L,\sigma} \frac{N(f_0^{L,\sigma})}{L} = 0 \qquad \forall \rho \tag{7}$$

This shows how unstable this type of condensation is to the introduction of any repulsive potential between the particles.

3. EFFECTS OF AN INTERACTION ON THE CONDENSATE

The idea of the proof of (7) is simple. Basically, one has to show that if (7) did not hold, the energy per particle would diverge when $L \to \infty$, in contradiction with the statement of Lemma 1 (see below).

Lemma 1. Take $b^L(x)$ a C^∞-function with support in $[0, L]$ and which is equal to $C^L/\sqrt{L}$ on the interval $[a, L-a]$ for some a and decreasing to zero outside this interval, with $\int_0^L |b^L(x)|^2\, dx = 1$. Assume, moreover,

$$\text{(i)}\quad \exists D_1 \text{ such that } |(b^L, h_1^L b^L)| \leqslant D_1 \qquad \forall L \tag{8a}$$

$$\text{(ii)}\quad \lim_{L\to\infty} \bar{f}_0^L(\beta) \equiv \lim_{L\to\infty} \frac{1}{L} \log \operatorname{Tr}_{\mathscr{H}_N^L} \exp(-\beta H_0^L)$$

exists for all $\beta > 0$. Then (8b)

$$\limsup_{L\to\infty} \frac{\omega^L(H^L)}{L} < \infty$$

Proof. Clearly

$$\bar{f}^L(\beta) \equiv \frac{1}{\beta L} \log \operatorname{Tr}_{\mathscr{H}_N^L} \exp(-\beta H^L) \leqslant \bar{f}_0^L(\beta) \tag{9}$$

Choose on the other hand $\psi^L = b^L \otimes \cdots \otimes b^L$. We have

$$\operatorname{Tr} \exp(-\beta H^L) \geqslant \exp[-\beta(\psi^L | H^L | \psi^L)]$$
$$\geqslant \exp -\beta \left[ND_1 + \frac{N(N-1)}{2L} (C^L)^2 a \right] \tag{10}$$

where $a = \int_{-\infty}^{+\infty} dx\, U(x)$.

Since $C^L \to 1$ and $N/L \to \rho$ as $L \to \infty$, it follows that for L sufficiently large,

$$-\beta(D_1 \rho + \rho^2 a/2) - \beta\varepsilon \leqslant \bar{f}^L(\beta) \leqslant \bar{f}_0^L(\beta) \tag{11}$$

Moreover, $\bar{f}^L_{(0)}(\beta)$ are convex functions of β. The lemma follows, noting that

$$-\omega^L \frac{H^L}{L} = \frac{d}{d\beta} \bar{f}^L(\beta) \geqslant \frac{\bar{f}^L(\beta) - \bar{f}^L(\beta - \delta)}{\delta} \qquad (\delta > 0)$$

and using the bounds (11). ∎

In the next proposition, it is shown that this lemma implies that condensation in a large class of functions is excluded.

Proposition 1. Suppose that conditions (8a)–(8b) of Lemma 1 hold. Consider a sequence of functions g^L with

$$\int_0^L |g^L(x)|^2\, dx = 1$$

and let

$$\tilde{g}_{L,\delta'}(x) = g^L(x)\, \chi_{[0, L^{\delta'}]}(x)$$

(χ_A being the characteristic function of the interval A). Then (i)

$$\lim_{L \to \infty} \omega^L \frac{N(\tilde{g}^{L,\delta'})}{L} = 0 \qquad \forall \delta' < \tfrac{1}{3} \tag{12}$$

(ii) If, moreover,

$$\int_x^{x+A} dx\, |g^L(x)|^2 \leqslant \frac{C^{ste}}{L^\alpha} \qquad \forall x \in [0, L - A]$$

[A as in (14)], then (12) holds for any $\delta' < \frac{1}{3}(1 + \alpha)$.

Proof. Note that

$$\frac{\omega^L(H^L)}{L} \geqslant -\varepsilon_0^L \frac{N}{L} + \frac{1}{2L}\int_0^L dx \int_0^L dy$$
$$\times U(x-y)\,\omega^L(a^*(x)\,a^*(y)\,a(y)\,a(x))$$
$$\geqslant -\varepsilon_0^L \frac{N}{L} + \frac{\varepsilon}{2L}\sum_{i=1}^{L^{\delta'}/A}\int_{(i-1)A}^{iA} dx \int_{(i-1)A}^{iA} dy$$
$$\times\,\omega^L(a^*(x)\,a^*(y)\,a(y)\,a(x)) \tag{13}$$

where A and ε are as in (4).

The rest of the proof then consists in finding a lower bound for

$$\frac{1}{L}\sum_{i=1}^{L^{\delta'}/A}\int_{(i-1)A}^{iA} dx \int_{(i-1)A}^{iA} dy$$
$$\times\,\omega^L(a^*(x)\,a^*(y)\,a(y)\,a(x)) \equiv \frac{1}{L}\sum_{i=1}^{L^{\delta'}/A} B_i^L$$

as a function of $\omega^L(N(\tilde{g}^{L,\delta'}))$. This bound would diverge as $L\to\infty$ if the proposition did not hold. Define g_i^L by

$$g_i^L(x) = g^L(x)\,\chi_{[(i-1)A,iA]}(x) \qquad 1 \leqslant i \leqslant L^{\delta'}/A$$

Then,

$$|\omega^L(a^*(g_i^L)\,a(g_j^L))|^4$$
$$\leqslant [\omega^L(a^*(g_i^L)\,a(g_i^L))]^2[\omega^L(a^*(g_j^L)\,a(g_j^L))]^2$$
$$\leqslant \omega^L(a^*(g_i^L)\,a(g_i^L)\,a^*(g_i^L)\,a(g_i^L))\,\omega^L(a^*(g_j^L)\,a(g_j^L)\,a^*(g_k^L)\,a(g_j^L)) \tag{14}$$

Moreover,

$$\omega^L(a^*(g_i^L)\,a(g_i^L)\,a^*(g_i^L)\,a(g_i^L))$$
$$= \omega^L(a^*(g_i^L)\,a^*(g_i^L)\,a(g_i^L)\,a(g_i^L)) + \|g_i^L\|^2\,\omega(a^*(g_i^L)\,a(g_i^L)) \tag{15}$$

Next, use the basic inequality $a^2+b^2 \geqslant 2ab$ to obtain

(i) $\omega^L(a^*(g_i^L)\,a(g_i^L))$

$$= \int_0^L dx \int_0^L dy\; g_i^L(x)\,\bar{g}_i^L(y)\,\omega^L(a^*(x)\,a(y))$$
$$\leqslant \int_0^L dx\,\omega^L(a^*(x)\,a(x)) \int_0^L dy\,|g_i^L(y)|^2$$
$$\leqslant N \tag{16a}$$

since $\int_0^L dx\, a^*(x)\, a(x)$ is equal to the number operator N^L and $\omega^L(N^L) = N$.

(iii) $\omega(a^*(g_i^L)\, a^*(g_i^L)\, a(g_i^L)\, a(g_i^L))$

$$\leqslant \int_{(i-1)A}^{iA} dx \int_{(i-1)A}^{iA} dy\, \omega^L(a^*(x)\, a^*(y)\, a(y)\, a(x)) \times \left(\int_{(i-1)A}^{iA} dz\, |g_i^L(z)|^2 \right)^2 \leqslant B_i^L \tag{16b}$$

Combining (14)–(16b) then yields

$$\sum_{i,j=1}^{L^{\delta'}/A} |\omega^L(a^*(g_i^L)\, a(g_j^L))|^4 \leqslant \left(\sum_{i=1}^{L^{\delta'}/A} B_i^L + \rho L \right)^2$$

Note finally that

$$\left| \sum_{i=1}^{n} X_i \right|^2 \leqslant n \sum_{i=1}^{n} |X_i|^2$$

which gives

$$\frac{A^6}{L^{6\delta'}} \left| \sum_{i,j=1}^{L^{\delta'}/A} \omega^L(a^*(g_i^L)\, a(g_j^L)) \right|^4 = \frac{A^6}{L^{6\delta'}}\, \omega(N(\tilde{g}^{L,\delta'}))^4 \leqslant \left(\sum_{i=1}^{L^{\delta'}/A} B_i^L + \rho L \right)^2 \tag{17}$$

or

$$\frac{1}{L} \sum_{i=1}^{L^{\delta'}/A} B_i^L \geqslant \frac{A^3}{L^{3\delta'+1}} [\omega^L(N(\tilde{g}^{L,\delta'}))]^2 - \rho \tag{17}$$

It is then easy to see that, if

$$\limsup_{L\to\infty} \frac{\omega^L(N(\tilde{g}^{L,\delta'}))}{L} > 0$$

then

$$\limsup_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L^{\delta'}/A} B_i^L = +\infty \qquad \text{if} \quad \delta' < \frac{1}{3}$$

or, by (13),

$$\limsup_{L\to\infty} \frac{\omega^L(H^L)}{L} = +\infty$$

in contradiction with Lemma 1. The second statement of the proposition follows by using the bound $\|g_i^L\|^2 \leqslant c^{ste}/L^\alpha$ in (16a)–(16b) instead of the bound $\|g_i^L\|^2 \leqslant 1$ as used to prove statement (i). ∎

The following corollaries are now immediate consequences of the preceding proposition.

Corollary 1. Consider the case $h_1^L = h_1^{L,\sigma}$. Then

$$\lim_{L\to\infty} \frac{\omega^L(N(f_0^{L,\sigma}))}{L} = 0 \qquad \forall \beta, \rho > 0$$

The conditions (8) are easily verified to hold. Therefore, by Proposition 1,

$$\lim_{L\to\infty} \omega^L(N(f_0^{L,0}\chi_{[0,L^{\delta'}]}))^{L^{-1}} = 0, \qquad 0 \leqslant \delta' < \tfrac{1}{3}$$

Using an estimate as in (16a), one verifies that this implies

$$\lim_{L\to\infty} \omega^L(N(f_0^{L,\sigma}))^{L^{-1}} = 0 \quad ∎$$

As mentioned before, this shows how unstable this type of Bose–Einstein condensation is. Even if we introduce an interaction as small as

$$\begin{aligned} U(x) &= \delta \qquad \text{if} \quad |x| \leqslant \varepsilon \\ &= 0 \qquad \text{else} \end{aligned}$$

we destroy the condensation in the level $f_0^{L,\sigma}$.

In fact, no condensation in any level should be expected in this case. This is shown rigorously in the van der Waals limit in the next section.

It should be noted that the result of Corollary 1 does not contradict the result stated in the introduction, namely that the introduction of a gap makes the condensate stable even when a small interaction is introduced.[6] Indeed, in that paper, the gap is of a different kind.

Proposition 1 can also be used to derive results in the case of an external field. Consider h_1^L of the following form:

$$h_1^L = -\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{x}{L}\right)^\alpha \tag{18}$$

with Dirichlet (or Neumann) boundary conditions on $[0, L]$.

The eigenfunctions and corresponding eigenvalues of h_1^L can be written as

$$f_n^L(x) = L^{-\alpha/2(\alpha+2)}\,\tilde{f}_n^L(x/L^{\alpha/(\alpha+2)})$$
$$\varepsilon_n^L = L^{-2\alpha/(\alpha+2)}\,\tilde{\varepsilon}_n^L$$

where $\tilde{f}_n^L$ are the eigenfunctions and $\tilde{\varepsilon}_n^L$ the eigenvalues of the operator

$$\tilde{h}_1^L = -\frac{1}{2}\frac{d^2}{dx^2} + x^\alpha$$

on the interval $[0, L^{2/\alpha+2}]$ with Dirichlet (Neumann) boundary conditions.

It can be shown that the free model shows condensation in the lowest energy level at high densities if $\alpha < 2$, i.e..,[17]

$$\lim_{L \to \infty} \omega_0^L \frac{N(f_0^L)}{L} \neq 0$$

Note that the length scale for the condensate is $L^{\alpha/(\alpha+2)}$. Using path integral techniques,[17] one can get estimates for the behavior of $\tilde{f}_n^L(x)$ for all x. Using the second statement o Proposition 1, these estimates are then sufficient to prove the following corollary.

Corollary 2. If the case of the one-particle Hamiltonians, we have

$$\lim_{L \to \infty} \frac{\omega^L(N(f_n^L))}{L} = 0 \qquad \forall n < \infty, \quad \forall \beta, \rho > 0 \quad \blacksquare \tag{18}$$

If the potential is sufficiently localizing (i.e., α close to zero), one might reasonably expect that there would remain some condensate. However, the preceding corollary shows that this condensate will have to sit on a length scale bigger than $L^{\alpha/(\alpha+2)}$.

4. INDEPENDENCE OF THE FREE ENERGY ON THE BOUNDARY CONDITION

Consider again the case of the one-dimensional interacting Bose gas with attractive boundary conditions [see (1)]. The two-body interaction U considered in this section will be assumed to satisfy conditions (4) and to be of positive type, i.e.,

$$\hat{U}(k) \equiv \int dk \exp ikx \; U(x) \geq 0 \qquad \forall k \in \mathbb{R} \tag{19}$$

The Hamiltonian for the interacting system will be written as

$$H_\lambda^{L,\sigma} = H_0^{L,\sigma} + \mathcal{U}_\lambda \tag{20}$$

where

$$\mathcal{U}_\lambda(x_1,..., x_N) = \lambda \sum_{1 \leqslant i \leqslant j \leqslant N} U(\lambda(x_i - x_j)), \qquad \lambda > 0$$

and where the index σ denotes, as before, the boundary conditions (1).

Let $f_\lambda^{L,0}(\rho)$ be the corresponding free energy, i.e.,

$$f_\lambda^{L,\sigma}(\rho) = -\frac{1}{\beta L} \log \operatorname{Tr}_{\mathscr{H}_N^L} \exp - \beta H_\lambda^{L,\sigma}$$

The limit $\lim_{\lambda\downarrow 0} \lim_{L\to\infty}$ is usually called the van der Waals limit and corresponds to a limit of very long-range interactions.

It is now shown that the free energy no longer depends on σ in this limit, in sharp contrast to both the free model and the hard core model studied by Buffet and Pulè.[11]

An interesting question is whether this result would also hold without taking the van der Waals limit. On physical grounds one expects the answer to this question to be yes. However, it is not clear how far the example Studied in Ref. 11 contradicts this conjecture. As a side result, it is also shown that, in the van der Waals limit, there is not only no condensation in the ground state (as shown in Proposition 1), but even no condensation in the generalized sense.

Theorem 1. (i) With the preceding conventions

$$\lim_{\lambda\downarrow 0} \lim_{L\to\infty} f_\lambda^{L,\sigma}(\rho) = f_0(\rho) + \frac{a\rho^2}{2}, \qquad a = \hat{U}(0)$$

where $f_0(\rho)$ denotes the limiting free energy of the free Bose gas with Neumann or Dirichlet boundary conditions. (ii) We have

$$\lim_{\varepsilon\downarrow 0} \lim_{\lambda\downarrow 0} \lim_{L\to\infty} \frac{1}{L} \sum_{\varepsilon_k^{L,\sigma} \leqslant \varepsilon} \omega_\lambda^{L,\sigma}(N(f_k^{L,\sigma})) = 0$$

where $\varepsilon_k^{L,\sigma}$ and $f_k^{L,\sigma}$ are as in (2) and where $\omega_\lambda^{L,\sigma}$ is as in (5b).

Proof. (i)

$$H_\lambda^{L,\sigma} \leqslant H_\lambda^{L,\infty} \qquad \forall \sigma \in \mathbb{R}$$

(the index ∞ denotes Dirichlet boundary conditions; this inequality has to be understood in the sense of Friedrich extensions). Therefore

$$\limsup_{L\to\infty} f_\lambda^{L,\sigma}(\rho) \leqslant \lim_{L\to\infty} f_\lambda^{L,\sigma}(\rho) \tag{21}$$

If $g^L(x)$ is defined as

$$g^L(x) = -\frac{1}{\beta L} \log \mathrm{Tr}_{\mathscr{H}_N^L} \exp - \beta\Big[x\varepsilon_0^{L,\sigma} N(f_0^{L,\sigma}) + \sum_{k\geqslant 1} \varepsilon_k^{L,\sigma} N(f_k^{L,\sigma}) + \mathscr{U}_\lambda\Big]$$

a standard argument about convexity leads to

$$f_\lambda^{L,\sigma}(\rho) - g^L(0) \geqslant -\frac{\sigma^2}{2L}\, \tilde{\omega}^L(N(f_0^{L,\sigma}))$$

where

$$\tilde{\omega}^L(A) = \left\{\mathrm{Tr}_{\mathscr{H}_N^L} A \exp - \beta\left[\sum_{k\geqslant 1} \varepsilon_k^{L,\sigma}\, N(f_k^{L,\sigma}) + \mathscr{U}_\lambda\right]\right\} \times \left\{\mathrm{Tr}_{\mathscr{H}_N^L} \exp - \beta\left[\sum_{k\geqslant 1} \varepsilon_k^{L,\sigma}\, N(f_k^{L,\sigma}) + \mathscr{U}_\lambda\right]\right\}^{-1}$$

Using Proposition 1, this becomes

$$\liminf_{L\to\infty} f_\lambda^{L,\sigma}(\rho) \geqslant \liminf_{L\to\infty} g^L(0) \tag{22}$$

Furthermore, it is proved in Ref. 18 that $\forall\lambda > 0$, $\forall\varepsilon > 0$, $\exists L_0$ such that $\forall L > L_0$, $\forall N$:

$$\mathscr{U}_\lambda(x_1,..., x_N) \geqslant \tfrac{1}{2}a(1-\varepsilon)N^2 - b\lambda N, \qquad b = U(0) \tag{23}$$

implying

$$\begin{aligned}\liminf_{L\to\infty} f_\lambda^{L,\sigma}(\rho) &\geqslant \lim_{L\to\infty} -\frac{1}{\beta L} \log \mathrm{Tr}_{\mathscr{H}_N^L} \exp - \beta\left[\sum_{k=1}^{\infty} \varepsilon_k^{L,\sigma}\, N(f_k^{L,\sigma})\right] + \frac{a\rho^2}{2} - b\lambda\rho \\ &= f_0(\rho) + \frac{a\rho^2}{2} - b\lambda\rho \end{aligned} \tag{24}$$

The result then follows by combining the bounds (21) and (24) and by taking the limit $\lambda \downarrow 0$.

(ii) The proof of this result is now completely analogous to the proof in Ref. 5.

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